ON THE GEOMETRY OF THE CIRCLE.

BY DR. VIRGIL SNYDER.

(Read before the American Mathematical Society, December 28, 1899.)

LET $x_1, x_2, \cdots, x_5$ be any five numbers which satisfy the homogeneous quadratic identity $\omega(x) \equiv 0$; these numbers may be taken as the homogeneous coördinates of the circle.

The linear geometry of the circle can be readily interpreted from the paper of the author* on Dupin's cyclides; and the corresponding theorems for the quadratic configurations from another paper.† In the latter, one of the numbers x_6 was given a restricted interpretation, that of representing all the points of space; and the resulting theorems all referred to cyclides. A set of similar theorems exists for the bicircular quartic curves when, in the geometry of the circle, one coördinate equated to zero represents the points of a plane. All the known theorems regarding these curves, as given by Casey,‡ Darboux,§ and Loria,‖ can be very easily derived, and a number of new ones which are not contained in these memoirs. Another specialization is that obtained by taking the lines of a plane as one of the fundamental complexes. This case has not been systematically treated.

Now suppose that any five complexes mutually in involution be taken as fundamental complexes. In general, none of these complexes is orthogonal, and no circle belongs to them all. The quadratic identity now becomes

$$\omega(x) \equiv \sum_{r=1}^{5} x_r^2 = 0.$$

These coördinates now represent circles, and not points, as in the memoirs quoted. This theorem results immediately: The curve of singularities of a general quadratic complex is

* V. Snyder, "On the determination of nodes in Dupin's cyclides," *Ann. of Math.*, vol. 11, p. 137.

† V. Snyder, "Geometry of some differential expressions," BULLETIN, vol. 4, p. 144.

‡ J. Casey, "On the bicircular quartics," *Trans. R. Irish Acad.*, vol. 24, p. 359.

§ G. Darboux, "Sur une classe remarquable de courbes et de surfaces," Paris, 1873.

‖ G. Loria, "Sur la géométrie analytique du cercle," *Quar. Jour., of Math.*, vol. 22, p. 44.

likewise curve of singularities of ∞^1 such complexes and is focal curve (or envelope) for five quadratic congruences.

The congruences whose envelopes coincide with the curve of singularities of $\sum_{\tau=1}^{5} a_\tau x_\tau^2 = 0$ are

$$x_\varkappa = 0, \qquad \sum_{\tau=1}^{5} \frac{x_\tau^2}{a_\tau - a_\varkappa} = 0 \quad (\tau \neq \varkappa).$$

The locus of centers of circles belonging to any congruence is easily found to be a bicircular quartic, hence:

The curve of singularities of a general quadratic complex is the envelope of circles which cut a fixed circle at a constant angle and whose centers lie on a bicircular quartic.

This curve can be generated in five-fold manner, and if φ_τ, $\varphi_\varkappa$ denote the angles of the τth and $\varkappa$th fundamental complexes, and $[\tau, \varkappa]$ the angle between their defining circles, then

$$\cos [\tau, \varkappa] = \cos \varphi_\tau \cdot \cos \varphi_\varkappa$$

for all values of τ, $\varkappa$ from 1 to 5.*

If any coördinate complex is orthogonal, its defining circle belongs to all the other coördinate complexes; if four of the coördinate complexes are orthogonal, the locus of centers in each congruence is a conic section and this case reduces to the one treated before.

By employing the method given on p. 150 of volume 4 of the BULLETIN, one easily obtains this theorem:

Eight tangents can be drawn to the curve of singularities parallel to any given line.

One cannot conclude that the curve is of class 8; for some particular cases arise in which the line at infinity is a multiple factor in the degraded curve, yet eight parallel tangents can be drawn to the remaining curve in any direction.

The order of the curve is 24, as may be found as follows: The problem is to find the envelope of the circle

$$(\mu - \nu)(x^2 + y^2) - 2\xi xt - 2\eta yt + (\mu + \nu)t^2 = 0,$$

subject to the three conditions

$$\omega(\xi, \eta, \lambda, \mu, \nu) = 0, \quad F_3(\xi, \eta, \lambda, \mu, \nu) = 0,$$
$$a\xi + b\eta + d\lambda + c\mu + f\nu = 0;$$

*See V. Snyder, "Ueber die linearen Complexe der Lie'schen Kugelgeometrie," Göttingen, 1895, p. 37.

or, in terms of the new coördinates, to find the envelope of $\sum_{\tau=1}^{5} s_\tau x_\tau = 0$, subject to the conditions

$$\sum_{\tau=1}^{5} a_\tau x_\tau^2 = 0, \quad \sum_{\tau=1}^{5} x_\tau^2 = 0, \quad \text{and (say)} \quad x_1 = 0.$$

Putting $x_1 = 0$ in all the forms, and differentiating, regarding x_5 (say) as constant, we obtain

$$\Delta = \begin{vmatrix} s_2 & s_3 & s_4 \\ a_2 x_2 & a_3 x_3 & a_4 x_4 \\ x_2 & x_3 & x_4 \end{vmatrix} = 0.$$

It now remains to eliminate x_2, x_3, x_4, x_5 between the four homogeneous equations

$$\sum_{\tau=2}^{5} s_\tau x_\tau = 0, (1,1); \Sigma x_\tau^2 = 0, (2,0); \Sigma a\, x_\tau^2 = 0, (2,0); \Delta = 0, (2,1).$$

In each parenthesis, the first number denotes the degree in which x_τ enters the equation, and the second the degree in s_τ. Now by the theory of elimination, the s_τ from the first equation will enter the x eliminant to the degree 2.2.2, and those from Δ to the degree 2.2.1; hence the s_τ appear to degree 12. But $s_\tau = 0$ is the cartesian equation of a circle, hence the curve of singularities is of degree 24.

The curve passes 12 times through each circle point; its equation contains 12 constants.

Straight lines, conics, circular cubics, and bicircular quartics, as well as their parallels, evolutes, and the involutes of their caustics by reflection, can be exhaustively classified from this standpoint; they all appear as degraded forms of the curve of singularities.

The lines drawn perpendicular to the asymptotes of the bicircular quartic (locus of centers), and cutting the fundamental circle of x_τ at the angle of the complex φ_τ, are all bitangents to the curve of singularities. The bicircular quartic has 4 asymptotes and there are two double tangents associated with each; hence each fundamental circle gives rise to 8 double tangents, all of which are tangent to a circle concentric with the given one. There are five such sets or forty in all (besides the line at infinity which may be a multiple tangent). The asymptotes of bicircular quartics, however, are all isotropic and perpendiculars to them must coincide with the lines themselves. They therefore fall to-

gether in pairs and pass through the centers of the five fundamental circles; hence:

The bitangents of the curve of singularities are all isotropic; one pair passes through the center of each fundamental circle.

The four points in which each circle intersects its associated quartic are foci of the curve of singularities; hence the latter has 20 single foci. There are 12 double foci, but they may not all be distinct.

By varying c in the equations

$$x_\kappa = 0, \qquad \sum_{\tau=1}^{5} \frac{x_\tau^2}{a_\tau - c} = 0,$$

a family of cosingular congruences is obtained, having a series of focal curves all of which pass through a finite number of fixed points. These curves are distributed into five distinct families. Two curves of the system pass through every point in the plane, the angle of intersection being a determinate function of φ_τ, φ_κ obtained by considering the tangent congruences of the complexes defining the curve at the given point. Each of the five sets contains a pair of circles as a degraded focal curve.

In a similar manner the following theorem may be proved:

The envelope of circles which cut a fixed circle orthogonally and whose centers lie on a curve of class n is a curve of order $2n$, which passes n times through each circle point.

Its foci and double tangents are easily determined as before.

A corresponding generalization can be made in the geometry of the sphere. The depiction of the Kummer surface in general sphere space is the envelope of spheres whose centers lie on a cyclide and which cut a fixed sphere at a constant angle. The same surface can be generated in six ways, though the loci of centers are not in general confocal nor are the fixed spheres orthogonal. Twelve tangent planes can be drawn parallel to any given plane. The cones of Kummer of bitangent planes which belong to the cyclide give rise to six developables of tangent planes to the imaginary circle at infinity. The lines of curvature on these developables are defined by pencils of spheres concentric with the fundamental spheres.*

This surface appears to be of order 64, contains the circle at infinity as a 32-fold line, and depends on 18 constants.

CORNELL UNIVERSITY,
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* See V. Snyder. "Lines of curvature on annular surfaces." *Amer. Jour. of Math.*, vol. 22, p. 96.

ISOMORPHISM BETWEEN CERTAIN SYSTEMS OF SIMPLE LINEAR GROUPS.

BY PROFESSOR L. E. DICKSON.

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1. In an article in the BULLETIN for July, 1899, giving the known finite simple groups, I made the conjecture that the simple quaternary hyperorthogonal group $HO(4, p^{2n})$ in the $GF[p^{2n}]$ was isomorphic with the second hypoabelian group $SH(6, 2^n)$, the orthogonal group $O(6, p^n)$, or the group $NS(6, p^n)$, according as p^n is of the form 2^n, $4l-1$, or $4l+1$ respectively. For the case* $p^n = 2$, and for the case† $p^n = 3$, I have proven the conjecture true by setting up abstract groups holoedrically isomorphic with the linear groups in question. The calculations were necessarily long, so that the method of procedure would scarcely be adapted to the case of general p^n. From the correspondence of generators established in these two special cases, I have been led to the short proof for the general case given in this paper. The proof is based upon the theory of the second compound of a linear homogeneous group, as developed in the BULLETIN for December 1898, and in the *Transactions* for January, 1900, pp. 91–96. In place of the hyperorthogonal group $HO(4, p^{2n})$, I employ the holoedrically isomorphic hyperabelian group‡ $HA(4, p^{2n})$, which contains as a subgroup the simple abelian group $A(4, p^n)$. The calculations appear to me to possess considerable elegance.

2. THEOREM.—*According as* -1 *is a square or a not-square in the* $GF[p^n]$ $(p > 2)$, $HA(4, p^{2n})$ *is holoedrically isomorphic with the group* $NS(6, p^n)$ *or* $O(6, p^n)$.

Corresponding to every substitution of the subgroup $A(4, p^n)$ there is a senary linear homogeneous substitution of the second compound H with the two invariants

$$(1) \quad \psi \equiv Y_{12}Y_{34} - Y_{13}Y_{24} + Y_{14}Y_{23}, \qquad Z \equiv Y_{12} + Y_{34}.$$

To the hyperabelian substitution, in which I belongs to the $GF(p^{2n})$,

$$I \equiv \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I^{-p^n} & 0 & 0 \\ 0 & 0 & I^{-1} & 0 \\ 0 & 0 & 0 & I^{p^n} \end{bmatrix}$$

* *Proc. of the Lond. Math. Soc.*, vol. 31, pp. 30–68.
† *Transactions*, vol. 1, No. 3 (July, 1900).
‡ *Proc. Lond. Math. Soc.*, vol. 31, pp. 30–68.

there corresponds in the second compound G of $HA(4, p^{2n})$, the substitution

$$I': \begin{cases} Y_{12}' = I^{-p^n+1}Y_{12}, & Y_{13}' = Y_{13}, & Y_{14}' = I^{p^n+1}Y_{14}, \\ Y_{23}' = I^{-p^n-1}Y_{23}, & Y_{24}' = Y_{24}, & Y_{34}' = I^{p^n-1}Y_{34}. \end{cases}$$

Introducing in place of Y_{12} and Y_{34} the new indices ($p > 2$)

$$(2) \qquad \xi_1 \equiv \tfrac{1}{2}(Y_{12} - Y_{34}), \quad \xi_6 \equiv \frac{-1}{2J}(Y_{12} + Y_{34}),$$

the reversion of which gives

$$(3) \qquad Y_{12} \equiv \xi_1 - J\xi_6, \quad Y_{34} \equiv -\xi_1 - J\xi_6,$$

the substitution I' takes the form

$$I'': \begin{cases} \xi_1' = \tfrac{1}{2}\theta\xi_1 + \tfrac{1}{2}J(I^{p^n-1} - I^{-p^n+1})\xi_6, \\ \xi_6' = \dfrac{1}{2J}(I^{p^n-1} - I^{-p^n+1})\xi_1 + \tfrac{1}{2}\theta\xi_6, \\ Y_{13}' = Y_{13}, \ Y_{14}' = I^{p^n+1}Y_{14}, \\ Y_{24}' = Y_{24}, \ Y_{23}' = I^{-p^n-1}Y_{23}, \end{cases}$$

$$(4) \qquad \theta \equiv I^{p^n-1} + I^{-p^n+1}.$$

Since $\theta^{p^n} = \theta$, θ belongs to the $GF[p^n]$. The coefficient

$$\tfrac{1}{2}J(I^{p^n-1} - I^{-p^n+1}),$$

whose p^nth power is $\frac{1}{2}J^{p^n}(I^{1-p^n} - I^{-1+p^n})$, belongs to the $GF[p^n]$ if and only if $J \equiv -J^{p^n}$, or $J^{p^n-1} = -1$. With this condition satisfied, the substitution I' belongs to the $GF[p^n]$ and leaves invariant the new form of ϕ, viz.,

$$\phi'' \equiv -\xi_1^2 + J^2\xi_6^2 - Y_{13}Y_{24} + Y_{14}Y_{23}.$$

By the relation just assumed to hold,

$$(J^2)^{p^n-1} = 1, \quad (J^2)^{(p^n-1)/2} = -1,$$

so that J^2 belongs to the $GF[p^n]$, but is a not-square in it. It follows* that the group G'' given by the extension of H''

* "Determination of the structure of all linear homogeneous groups in a Galois field which are defined by a quadratic invariant," *Amer. Jour. of Math.*, vol. 21, pp. 193–256.

(H when expressed in the indices ξ_1, ξ_6, Y_{13}, Y_{14}, Y_{23}, Y_{24}) by the substitution I'' is a subgroup of the group Q of senary linear substitutions in the $GF[p^n]$ which leaves ψ'' invariant. The latter has a subgroup Q'' of index 2 which contains all the substitutions* Q_{16}. Further, Q'' is extended to Q by any substitution O_{16} which is not of the form Q_{16}. We proceed to prove that $G'' \equiv Q''$. In order that H'' shall contain the substitution

$$h: \quad Y_{14}' = I^{p^n+1} Y_{14}, \quad Y_{23}' = I^{-p^n-1} Y_{23},$$

it is necessary and sufficient that I^{p^n+1} be a square in the $GF[p^n]$, or

$$I^{(p^n+1)(p^n-1)/2} = +1,$$

i. e., that I be a square in the $GF[p^{2n}]$. If this condition be satisfied, G'' contains the product $K \equiv I''h^{-1}$, viz.,

$$\xi_1' = \tfrac{1}{2}\theta\xi_1 + \tfrac{1}{2}J(I^{p^n-1} - I^{-p^n+1})\xi_6,$$

$$\xi_6' = \frac{1}{2J}(I^{p^n-1} - I^{-p^n+1})\xi_1 + \tfrac{1}{2}\theta\xi_6,$$

having determinant unity and leaving invariant $-\xi_1^2 + J^2\xi_6^2$. Let

$$(5) \quad \alpha \equiv \tfrac{1}{2}\left(I^{\frac{p^n-1}{2}} + I^{\frac{-p^n+1}{2}}\right), \quad \beta \equiv \tfrac{1}{2}J\left(I^{\frac{p^n-1}{2}} - I^{\frac{-p^n+1}{2}}\right)$$

We readily find that

$$(6) \quad \alpha^2 - \frac{1}{J^2}\beta^2 = 1, \quad 2\alpha\beta = \frac{J}{2}(I^{p^n-1} - I^{-p^n+1}),$$

$$2\alpha^2 - 1 = \tfrac{1}{2}(I^{p^n-1} + I^{-p^n+1}).$$

Hence $K = Q_{1,6}^{\alpha,\beta}$, α and β being marks of the $GF[p^n]$. Inversely, if α and β be any set of solutions in the $GF[p^n]$ of the first relation (6), we can determine a square I in the $GF[p^{2n}]$ which satisfies the last two relations (6). In fact, from them,

$$I^{p^n-1} = 2\alpha^2 - 1 + \frac{2\alpha\beta}{J} = \left(\alpha + \frac{\beta}{J}\right)^2, \quad I^{-p^n+1} = \left(\alpha - \frac{\beta}{J}\right)^2.$$

The second follows from the first, since

$$\left(\alpha + \frac{\beta}{J}\right)^2\left(\alpha - \frac{\beta}{J}\right)^2 = \left(\alpha^2 - \frac{\beta^2}{J^2}\right)^2 = 1.$$

* For the notations, see *Amer. Jour. of Math.*, vol. 21, p. 207.

To show that the first relation can be solved for a square I in the $GF[p^{2n}]$, we note that

$$\left(\alpha+\frac{\beta}{J}\right)^{p^n+1}=\left(\alpha+\frac{\beta}{J^{p^n}}\right)\left(\alpha+\frac{\beta}{J}\right)=\alpha^2-\frac{\beta^2}{J^2}=1.$$

We have therefore shown that G'' contains every $Q_{1,\,6}^{\alpha,\,\beta}$. For I a not-square in the $GF[p^{2n}]$, the substitution I'' is the product of an $O_{1,\,6}^{\alpha,\,\sigma}$ (not a Q_{16}) by h, neither of which separately belongs to G''. By introducing new indices ξ_{23}, $\xi_{\bar{2}\bar{3}}$ in place of Y_{14}, $Y_{\bar{2}\bar{3}}$, we may give h the form $O_{2,\,\bar{3}}^{\kappa,\,\lambda}$ (not a Q_{23}). It follows that $G''\equiv Q'$. In order to obtain the quotient group $HA(4, p^{2n})$, we consider as identical the products S and $ST\equiv TS$, where T is a hyperabelian substitution of determinant unity multiplying every index by τ, so that

$$\tau^4=1,\quad \tau^{p^n+1}=1.$$

Hence, if $p^n=4l+1$, the second compound G'' of $HA(4, p^{2n})$ is simple, and therefore holoedrically isomorphic with $NS(6, p^n)$. If $p^n=4l-1$, G'' has a maximal invariant subgroup of order two, generated by the substitution changing the signs of all six indices, the quotient group being $O(6, p^n)$.

3. THEOREM.—*The simple groups $HA(4, 2^{2n})$ and $SH(6, 2^n)$ are holoedrically isomorphic.*

Let λ be a mark of the $GF[2^n]$ such that the equation

$$\sigma^2+\sigma+\lambda^2=0 \tag{6}$$

is irreducible in the $GF[2^n]$. The condition on λ for the irreducibility of (6) is readily given (*American Journal of Mathematics*, l. c., p. 224) and can always be satisfied. Its roots are then σ and σ^{2^n} where

$$\sigma+\sigma^{2^n}=-1. \tag{7}$$

We apply to the substitution I' of §2 the transformation of indices

$$Y_{12}\equiv\lambda^{\frac{1}{2}}\xi_1-\sigma\lambda^{-\frac{1}{2}}\xi_6,\quad Y_{34}\equiv\lambda^{\frac{1}{2}}\xi_1-\lambda^{\frac{1}{2}}\sigma^{-1}\xi_6, \tag{8}$$

which gives reciprocally, for $p=2$,

$$\xi_1\equiv\lambda^{\frac{1}{2}}\sigma^{-1}Y_{12}+\lambda^{-\frac{1}{2}}\sigma Y_{34},\quad \xi_6=\lambda^{\frac{1}{2}}(Y_{12}+Y_{34}). \tag{9}$$

Multiplying I' by h^{-1} (h defined as in §2), we obtain the substitution:

$$T':\quad Y_{12}'=\tau Y_{12},\quad Y_{34}'=\tau^{-1}Y_{34}\qquad(\tau=I^{-2^n+1}).$$

Expressed in terms of the new indices ξ_1, ξ_6 defined by (9), T becomes

$$T'': \begin{cases} \xi_1' = (\lambda^3\sigma^{-1}\tau + \sigma\tau^{-1})\xi_1 - \lambda(\tau + \tau^{-1})\xi_6, \\ \xi_6' = \lambda(\tau + \tau^{-1})\xi_1 - (\lambda^3\sigma^{-1}\tau^{-1} + \sigma\tau)\xi_6. \end{cases}$$

We readily prove that the coefficients of T'' belong to the $GF[2^n]$. Since $\tau \equiv I^{-2^n+1}$, we find that $\tau + \tau^{-1}$ equals its 2^nth power and is therefore in the $GF[2^n]$. Furthermore, by (6),

$$\lambda^2\sigma^{-1}\tau + \sigma\tau^{-1} \equiv \tau + \sigma(\tau + \tau^{-1}) \pmod 2.$$

The latter equals its 2^nth power since, by (7), we have

$$\tau^{2^n} - \tau + (\sigma^{2^n} - \sigma)(\tau + \tau^{-1}) = (\tau + \tau^{-1})(\sigma^{2^n} - \sigma + 1)$$
$$[\equiv 0 \pmod 2.$$

Changing τ into τ^{-1}, it follows that $\lambda^3\sigma^{-1}\tau^{-1} + \sigma\tau$ belongs to the $GF[2^n]$.

By the transformation (8), ψ becomes, on applying (6),

$$\psi_1 \equiv \lambda(\xi_1^2 + \xi_6^2) + \xi_1\xi_6 - Y_{13}Y_{24} + Y_{14}Y_{23}.$$

Hence the group I'' given by the extension of the group H'' by T'' is a subgroup of $SH(6, 2^n)$. Evidently H'' contains all the substitutions of $SH(6, 2^n)$ which leave ξ_6 (as well as ψ_1) invariant. But $SH(6, 2^n)$ is given by the extension of H'' by the substitutions *

$$O_{1,6}^{\alpha,\beta}: \begin{cases} \xi_1' = a\xi_1 + \lambda(a + \beta)\xi_6 \\ \xi_6' = \lambda(a + \beta)\xi_1 + \beta\xi_6 \end{cases} \quad [a\beta + \lambda^2(a^2 + \beta^2) = 1].$$

The totality of substitutions $O_{1,6}^{\alpha,\beta}$ is identical with the totality of substitutions T'' when τ takes all possible values for which $\tau^{2^n+1} = 1$. In fact, if we set

$$a = \lambda^3\sigma^{-1}\tau + \sigma\tau^{-1}, \quad \beta = \lambda^3\sigma^{-1}\tau^{-1} + \sigma\tau,$$

we get by (6),

$$a + \beta = \tau + \tau^{-1}, \quad a\beta + \lambda^2(a^2 + \beta^2) = 1.$$

Hence every T'' is of the form $O_{1,6}^{\alpha,\beta}$. Inversely, if we set

$$\tau = a + \sigma(a + \beta), \quad \text{whence} \quad \tau^{-1} = \beta + \sigma(a + \beta),$$

we see that every $O_{1,6}^{\alpha,\beta}$ is of the form T''.

* *Amer. Jour. of Math.*, vol. 21, pp. 225–233. For uniformity of notation, I here use $O_{1,6}^{\alpha,\beta}$ in place of $O_1^{\alpha,\beta}$ of p. 226, replacing η_1 by ξ_6.

It follows that I''' is identical with $SH(6, 2^n)$. But I'' is holoedrically isomorphic with $G_{p=2}$ and therefore with $HA(4, 2^{2n})$, whose second compound is $G_{p=2}$.

4. NOTE.—It appears that the quaternary transformation group which naturally corresponds to the finite group $HO(4, p^{2n})$ is *not* continuous.

THE UNIVERSITY OF TEXAS,
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THE HESSIAN OF THE CUBIC SURFACE. II.

BY DR. J. I. HUTCHINSON.

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THE aim of the following paper is to extend the results of a previous article on the same subject (BULLETIN, March, 1899, p. 282) by determining all the quintic and sextic curves on the Hessian of the cubic surface, and giving some theorems connected with them, and with the quartic curves already determined.

I will write the equation of the Hessian in the form

$$F \equiv xyzu + wyzu + wxzu + wxyu + wxyz = 0,$$

where w, x, y, z, u are connected by the relation

$$aw + bx + cy + dz + eu = 0,$$

in which a, b, c, d, e are arbitrary constants.

As already shown, the surface F contains three classes of biquadratic curves, viz.:

α_1. A class containing 15 families which lie on 30 families of cones, all the cones of the same family cutting F in two lines and tangent along a third.

α_2. A class containing 30 families of curves lying on 30 families of cones tangent to F along two lines.

α_3. A class containing 15 families of curves determined by as many families of quadrics each intersecting F in a gauche quadrilateral, and by 30 families of quadrics each meeting F in two lines and a conic.

Consider the family of α_1 determined by the cones

$$(1) \qquad \begin{aligned} A_1 &\equiv x(w + y) + \lambda wy = 0, \\ A_1' &\equiv x(z + u) + (1 - \lambda)zu = 0. \end{aligned}$$

The plane $w + y = 0$ is tangent to all of the cones A_1 and hence is a double tangent plane for each biquadratic curve of the family. The two points of tangency with each curve form pairs of points in an involution on the line wy. Similarly, the plane $z + u = 0$ is a double tangent plane for each curve of the family, the points of contact forming an involution on the line zu.

All the curves of the family pass through the four nodes of F which lie on the lines wx and xy excluding their point of intersection (or we may say, the four which lie in the plane x excluding the vertices of A_1 and A_1'). These are the only points common to two curves of the same family.

The 15 families of α_1 can be grouped in 5 sets of 3 each, any one set having the property that a cubic surface can be passed through three curves arbitrarily chosen, one from each family of the set. For example, with the family determined by equations (1) are associated the two others obtained by the permutations (wz) and (wu). The family of cubic surfaces which intersects F in these three families of curves has the equation

$$A_1[x + \lambda'(1 - \lambda'')z + \lambda''(1 - \lambda')u] + A_1'[(1 - \lambda')(1 - \lambda'')w + x + \lambda'\lambda''y] = 0,$$

since the left member is unaltered by the permutations $(wz)(\lambda\lambda')$ and $(wu)(\lambda\lambda'')$.

Consider the family α_3 determined by the intersection of the quadrics

$$(2) \qquad \begin{aligned} A_3 &\equiv wy + \lambda zu = 0, \\ A_3' &\equiv wx + wy + xy - \lambda x(z + u) = 0. \end{aligned}$$

Each curve of the family passes through the same four nodes of F which form the points of intersection of the curve of the family (1) of α_1. These two families of α_1 and α_3 are also associated in such a way that they form the complete intersection with F of the family of quadrics

$$A_1 - \lambda' A_1' \equiv (\lambda - 1)A_3 + A_3' = 0.$$

The four points in which a given curve of the family (2) meets the lines wz, wu, yz, yu lie in a plane whose equation is

$$y + w - \lambda(z + u) = 0.$$

The plane $w + u = 0$ intersects A_3 in two lines one of which is the line wu. The plane is accordingly tangent to

A_3 in the point whose coördinates are $w = u = 0$, $y = \lambda z$. These coördinates also satisfy the equation $A_3' = 0$. Hence, the plane $w + u$ touches each biquadratic of the family in that point of the line wu where it is tangent to the quadric A_2 containing the given curve. Similarly for the planes $w + z$, $y + z$, $y + u$.

By means of the permutations (wz) and (wu) applied to the equations (2) we determine two other families which together with the first are so related that a cubic surface can be passed through three curves arbitrarily chosen, one from each family. The equation of the cubic is

$$A_3[\lambda''w + (1 + \lambda')(1 + \lambda'')x + \lambda'y] - A_3'[\lambda''w + \lambda'y - \lambda'\lambda''z - u] = 0,$$

since the left member is unaltered by the permutations $(yu)(\lambda\lambda')$ and $(wu)(\lambda\lambda'')$.

The curves (1) of α_1, and (2) of α_2 have in common the property that they meet any of the conics $\{xw\}$ (in the plane $w + x = 0$), $\{xy\}$, $\{xz\}$, $\{xu\}$ in pairs of points forming an involution.

Any curve of (1) intersects any curve of (2) in 8 points, on account of the identity

$$A_1' - \mu A_1' + (1 - \lambda)A_3 - A_3' \equiv 0,$$

where A_1, A_1' contain the parameter λ, and A_3, A_3' contain μ. But since

$$(1 - \lambda)A_1 + \lambda\mu A_1' - \lambda(1 - \lambda)A_2 \equiv x[(1 - \lambda)(w + y) + \lambda\mu(z + u)],$$

it follows that the four movable points of intersection lie in the plane

$$(1 - \lambda)(w + y) + \lambda\mu(z + u) = 0,$$

the remaining four being in the plane $x = 0$ as already noticed.

Consider the family of α_2 determined by the cones

$$(3) \qquad \begin{aligned} A_2 &\equiv \lambda w^2 + wx + wy + xy = 0, \\ A_2' &\equiv \lambda(zu + wz + wu) + (x + y)(z + u) = 0. \end{aligned}$$

Two curves of the same family do not intersect.

The planes $w + x$, $w + y$, and $z + u$ are double tangent planes to each curve of the family, the pairs of points so de-

termined forming involutions on the lines wx, wy, and zu respectively.

The four lines xz, yz, xu, yu are met by each curve in one point each, the four points so determined lying in the plane

$$\lambda w + x + y = 0.$$

If we apply to equations (3) the permutations (wxz) and (wzx), we obtain two other families of curves which together with (3) form the complete intersections of F and a family of cubic surfaces whose equation is

$$A_2[\lambda' x + y + \lambda''(1-\lambda')z - (1-\lambda')(1-\lambda'')u] + A_2'[(\lambda'\lambda'' - \lambda'' + 1)w + \lambda'\lambda'' x + y + \lambda'' z] = 0.$$

The function in the left member is unchanged by the permutation $(wxz)(\lambda\lambda'\lambda'')$. By means of the permutations (xy), (zu), and $(xy)(zu)$ which leave A_2 and A_2' unchanged we see that the family (3) is also associated in a similar manner with three other pairs of families.

There are four classes of quintic curves, all of the first species according to Salmon's classification. (Geometry of three dimensions, 4th edition, p. 318.)

β_1. Ten families of quintics formed by ten families of quadric cones passing through three lines of F which meet in a point. Such are the cones

$$B_1 \equiv \mu_1 xy + \mu_2 wy + \mu_3 wx = 0. \tag{4}$$

The same family of quintics is also determined by the partial intersection of F and the three families of cubic surfaces

$$\Gamma_1 \equiv \mu_1 xy(z+u) + zu[(\mu_1 - \mu_3)x + (\mu_1 - \mu_2)y] = 0,$$
$$\Gamma_2 \equiv \mu_2 wy(z+u) + zu[(\mu_2 - \mu_3)w + (\mu_2 - \mu_1)y] = 0,$$
$$\Gamma_3 \equiv \mu_3 wx(z+u) + zu[(\mu_3 - \mu_2)w + (\mu_3 - \mu_1)x] = 0,$$

since

$$\mu_1 F \equiv w\Gamma_1 + zuB_1, \quad \mu_2 F \equiv x\Gamma_2 + zuB_1, \quad \mu_3 F \equiv y\Gamma_3 + zuB_1.$$

If Γ_i' denote the function Γ_i with μ_k' written in the place of μ_k, it is easy to see that the two curves whose parameters are μ_k and μ_k' lie on the cubic surface Γ, where

$$\Gamma \equiv \mu_1'\Gamma_1 + \mu_2'\Gamma_2 + \mu_3'\Gamma_3 \equiv \mu_1\Gamma_1' + \mu_2\Gamma_2' + \mu_3\Gamma_3'.$$

For $\mu_k' = \mu_k$ we obtain a family of cubic surfaces inscribed to F along the quintics of the family (4) and the line zu.

By taking $\mu_1 = 0$ we see that this family of quintics includes three families of plane cubics, and by taking $\mu_1 = \mu_2$ we find that it also includes three families of biquadratics of a_1.

Every curve of the family passes through the seven nodes lying on the lines wx, wy, and xy. Any two curves intersect in two other points whose joining line is evidently a common generator of the two cones on which the two curves lie.

Through any two curves of β_1, whether belonging to the same family or not, a cubic surface can be passed which meets F elsewhere in two lines.

β_2. The next class of quintics contains sixty families determined by sixty families of cones, each passing through intersecting lines of F and tangent along one of them. The equation

$$B_2 \equiv ay^2 + \beta wy + \gamma x(w + y) = 0 \tag{5}$$

leads to a family of this class which is also determined by the cubic surfaces

$$G \equiv ay(xz + xu + zu) + (a - \beta + \gamma)xzu + \gamma x^2(z + u) = 0,$$

since

$$(a - \beta)F \equiv G(w + y) - B_2(xz + uz + xu).$$

Every curve of the family passes through the three nodes which lie on the line xy. Any two of the curves intersect in two other points.

β_3. The third class contains sixty families determined by sixty families of quadrics passing through two non-intersecting lines of F and a third line cutting both. The two equations

$$\begin{aligned} B_2 &\equiv y(\lambda w + \mu x) + wz = 0, \\ H_3 &\equiv xu(w + z) - (\lambda w + \mu x)(yu + xu + xy) = 0, \end{aligned} \tag{6}$$

determine such a family, since

$$F \equiv yH + B_2(yu + xu + xy).$$

Every curve of the family passes through the five nodes situated on the lines wz, wy, and yz excluding the two points of intersection. Any two curves have two other points of intersection.

If we apply to equations (6) the permutations $(wx)(zu)$, $(yz)(xu)$, $(wz)(yu)$, $(wu)(xy)$, we obtain together with (6) five families of β_3 having the property that if two curves be

taken from any two different families of the set, they will lie on a cubic surface which either passes through a conic, or is tangent to F along a line. The 60 families of β_3 group themselves into 12 sets of 5 of the like character.

β_4. The fourth class contains 30 families determined by 30 families of quadrics meeting F in a line and a conic. Such a family is given by the equations

$$B_4 \equiv zu + yu + yz - (w + x)(\lambda y + \mu z) = 0,$$
$$J \equiv \lambda wxy + \mu wxz + yzu = 0,$$

since $$F \equiv wxB_4 + (w + x)J.$$

Any two curves of the family intersect in five points, three of which are the nodes contained on the line yz.

There are six different classes of sextic curves on the Hessian.

γ_1. The first class contains 30 families, each of which is determined by the quadrics passing through two intersecting lines of F. The curves determined by the equation

$$C_1 \equiv wp + xy = 0,$$

where $$p = \alpha w + \beta x + \gamma y + \delta z + \varepsilon u,$$

also lie on the cubic surfaces

$$K_1 \equiv -zup + xyz + xyu + xzu + yzu = 0,$$

since $$F \equiv zuC_1 + wK_1.$$

Any two curves of this family with parameters $\alpha, \cdots$ and $\alpha_1, \cdots$ intersect in the four nodes lying on the lines wx and wy, excepting their common point, and in six other points lying on a conic in the plane $p - p_1 = 0$.

A second family generated by the equations

$$\overline{C}_1 \equiv wp' + zu = 0,$$
$$\overline{K}_1 \equiv -xyp' + xyz + xyu + xzu + yzu = 0$$

is associated with the preceding in such a way that a curve from each family lies on the cubic surface

$$K \equiv p'C_1 - K_1 \equiv p\overline{C}_1 - \overline{K}_1 = 0.$$

γ_2. The second class, containing 10 families, is determined by quadrics which touch F along a line. Such a family is

$$C_2 \equiv (w + x)p + wx = 0,$$

also determined by the surfaces

$$K_2 \equiv -yzu + p(yz + yu + zu).$$

In this case we have

$$F \equiv C_2(yz + yu + zu) - (w + x)K_2.$$

γ_3. The third class also contains 10 families. The quadric

$$C_2 \equiv (w + x)p + yz + yu + zu = 0,$$

which meets F in a conic, and the cubic surface

$$K_3 \equiv wxp - yzu = 0$$

determine such a family, since

$$F \equiv wxC_2 - (w + x)K_3.$$

Any two curves of the same family in either γ_2 or γ_3 intersect in six points lying on a conic.

A curve from each of the two families of γ_2 and γ_3 just mentioned lies on the cubic surface

$$K \equiv C_2p' + K_3 \equiv C_2'p + K_3'.$$

γ_4. The fourth class contains 15 families, each determined by the intersection with F of a family of quadrics such as

$$\alpha wy + \beta wz + \gamma xy + \delta xz = 0,$$

passing through two non-intersecting lines wx, yz. A sextic of this class cannot lie on a cubic surface.

γ_5. The fifth class of sextics contains 60 families. These curves are formed by the partial intersection of two cubic surfaces such as

$$\Delta_1 \equiv y(wx + wz + xz) + z(awx + \beta xz + \gamma wz) = 0,$$
$$\Delta_2 \equiv u(awx + \beta xz + \gamma wz) - wx(y + u),$$

which intersect in the three lines wx, wz, xz, and a sextic on F, since

$$F \equiv u\Delta_1 - z\Delta_2.$$

γ_6. The sixth class contains a single family, the Steiner sextics, determined by the equation

$$S \equiv xxyz + \lambda wyz + \mu wxz + \nu wxy = 0,$$

or equally well by four other equations referred to the four other fundamental tetrahedra.

Let s and s' be any two curves of the family and let $T = 0$, $T' = 0$ be the two cubic surfaces which touch F along s and s' respectively. Also let $H = 0$ be the cubic

surface containing both s and s'. Then an identity of the form

$$QF \equiv TT' + H^2 \tag{I}$$

necessarily exists, where Q is a function of the second degree. A course of reasoning similar to that used by Humbert in connection with the Kummer surface enables us to draw a number of interesting conclusions from (I).

The two curves s and s' intersect in four points p_1, p_2, p_3, p_4 (besides the 10 nodes of F through which every curve of the family passes). The five surfaces Q, F, T, T', H are mutually tangent at each of these four points.

The surface T intersects H in s and a twisted cubic c. It is evident that c lies on Q and that T touches Q along this cubic. Similarly, T' touches Q along a cubic c' which also lies on H.

The 24 points of intersection of Q, H, and F are double points of the surface $TT' = 0$. Among these occur the points p_i, each counted four times. The remaining eight points, since they lie on the curves s and s', are not points of tangency of T and T'. They must therefore be nodes on either T or T'. It is evident that half of them are on the one, and half on the other surface. Let a_1, a_2, a_3, a_4 be the four nodes on T. The joining line of any two of these points lies entirely on T, and is accordingly tangent to F in a third point. Hence, *the tetrahedron whose vertices are the nodes of the cubic surface T which touches F along s is inscribed to F as to its vertices and circumscribed to F as to its edges.* There are evidently ∞^3 such tetrahedra.

The four points a_i, and the points p_i counted twice, form the complete intersection of the cubic c with F.

The cubic surfaces S and S' defining the sextics s and s' intersect in six lines and a twisted cubic k. The latter passes through the four points p_i and the four vertices of the tetrahedron of reference. Conversely, every twisted cubic k passing through the vertices of the tetrahedron of reference intersects F in four remaining points p_i, which form the four points of intersection of two sextics s and s' (and hence are the common points of a singly infinite system of sextics.)

The Hessian F is invariant for the birational transformation T_5 defined by the equations

$$aw' : bx' : cy' : dz' = \frac{1}{w} : \frac{1}{x} : \frac{1}{y} : \frac{1}{z}.$$

This transformation interchanges the family of cubics k with the lines of space, and the sextics s with the plane

quartics of F.* Four other transformations of a similar character may be obtained by starting from any of the other four tetrahedra contained in the fundamental pentahedron. These are

$$T_1: \quad bx' : cy' : dz' : ew' = \frac{1}{x} : \frac{1}{y} : \frac{1}{z} : \frac{1}{u},$$

$$T_2: \quad aw' : cy' : dz' : eu' = \frac{1}{w} : \frac{1}{y} : \frac{1}{z} : \frac{1}{u},$$

$$T_3: \quad aw' : bx' : dz' : eu' = \frac{1}{w} : \frac{1}{x} : \frac{1}{z} : \frac{1}{u},$$

$$T_4: \quad aw' : bx' : cy' : eu' = \frac{1}{w} : \frac{1}{x} : \frac{1}{y} : \frac{1}{u}.$$

These five operations, each of period 2, generate a group G of infinite order, since the operation T_iT_j is of infinite period. As far as the points of F are concerned, each transformation T_i has exactly the same effect. In other words, *there exists a subgroup of index* 2 *under the group* G *for which each point of* F *is unchanged in position.*

Since a plane quartic has 28 bitangents it follows that the sextic s has 28 bitangent cubics k out of each of the five families of such cubics. Suppose the plane of the quartic q corresponding to s to be rotated about one of the bitangents of the curve. The intersections of the moving plane with F will form a single infinity of plane quartics having the same line for bitangent. Hence, for a given sextic s there are 28 subfamilies of ∞^1 sextics, each family being tangent to s in a pair of points, or in other words, the four points p_i fall together in pairs 28 times on a given sextic.

Two sextics do not in general touch each other at a node of F. There are ∞^2 sextics which touch each other and have a given generator of the tangent cone at the node for a common tangent line.

Suppose that we consider the ∞^1 of these curves which are also mutually tangent at a second node of F. If the joining line of the first and second nodes is a line of F, then these curves are also mutually tangent in two other nodes of F, the four nodes together forming the vertices of one of the five fundamental tetrahedra. If the first and second nodes do not have a line of F for joining line then the two remaining points of intersection determine a line which meets two of the lines of F. For example, if the two nodes be

* Cf. Salmon, Geom. of Three Dim., 4th ed., p. 495.

wxu and *yzu* then the joining line of the two remaining points of intersection of the sextics so determined will meet F elsewhere in the two lines *wx* and *yz*.

CORNELL UNIVERSITY,
February, 1900.

NOTE ON THE GROUP OF ISOMORPHISMS.

BY DR. G. A. MILLER.

(Read before the American Mathematical Society, February 24, 1900.)

LET $s_1, s_2, \cdots, s_g$ represent all the operators of a group G and let $t_\alpha s_\alpha$ correspond to s_α $(\alpha = 1, 2, \cdots, g)$ in any given simple isomorphism of G with itself. It is evident that t_α is some operator of G. When G is abelian these t_α's must constitute a group T which is isomorphic with G.* In this isomorphism, t_α evidently can not be the inverse of s_α unless $s_\alpha = 1$. As this condition is sufficient as well as necessary, we have

THEOREM I.—*Every simple isomorphism of an abelian group A with itself may be obtained by* 1° *making A isomorphic with one of its subgroups or with itself in such a manner that no operator corresponds to its inverse, and* 2° *making each operator of A correspond to itself multiplied by the operator which corresponds to it in the given isomorphism.*

The simplest case that can present itself is the one in which the subgroup of G, which corresponds to identity of T in the given isomorphism between G and T, includes T. The resulting simple isomorphism of G with itself must correspond to an operator in the group of isomorphisms of G, whose order is equal to the operator of highest order in T. When the order of T is an odd prime number p, or the double of an odd prime, only one other case can present itself; viz, the case in which T corresponds to itself, or to its subgroup of an odd prime order, in the given isomorphism between G and T. The resulting simple isomorphism of G with itself may clearly correspond to a cyclical group of order $p - 1$, or to any one of its subgroups in the group of isomorphisms of G. These results lead to the following

* When G is non-abelian, these t_α's need not constitute a group, as can be seen from the simple isomorphisms of the symmetric group of order 6 with itself.

THEOREM II.—*If we make an abelian group A simply isomorphic with itself by multiplying its operators by the operators of a subgroup whose order is an odd prime number p or the double of this prime number, the resulting simple isomorphism of A with itself will correspond to an operator of order p, $2p$, or $\frac{p-1}{a}$ (a being any division of $p-1$) in the group of isomorphisms of A.**

The determination of all the possible orders of the corresponding operators in the group of isomorphisms of any abelian group, when T is a given subgroup, seems to be a problem of considerable difficulty. When the order of T is small the number of cases that have to be considered is also small. In addition to the orders included in the theorems given, we have the following when the order of T does not exceed 8: If T is the cyclical group of order four, the resulting isomorphism may correspond to an operator of order two in the group of isomorphisms, and when T is the non-cyclical group of this order, it may correspond to operators of orders 3 and 4. When T is the cyclical group of order 8, the orders of these operators may be 2, 4, and 8; when T is the direct product of the cyclical group of order 4 and an operator of order 2, the orders of the corresponding operators in the group of isomorphisms may be 2 and 4; finally, when T is the direct product of three operators of order 2, the given operators may be of orders 2, 3, 4, 6, and 7. While all of the possible cases for a given T may present themselves in the same group it is evident that this does not always happen.

For the sake of illustration we may consider the group of isomorphisms of the group of order 8 which is the direct product of three operators of order 2.† Each of its 7 subgroups of order 4 leads to three operators of order two, according to the second sentence below Theorem I. We thus obtain the 21 operators of order 2 of the required group of isomorphisms when we consider all the possible instances in which the order of T is 2. If the order of T is 4, two cases present themselves—in one case just two of the operators of T (including identity) correspond to operators of T, and in the other case each one of the operators of T corresponds to some operator of T. The former case leads to the 42 operators of order 4 and the latter to 56 operators of order 3 of the required group of isomorphisms. Finally, we obtain 48 operators of order 7 when we consider all the pos-

* Cf. *Comptes rendus*, vol. 130, 1900, p. 316.
† Cf. Moore, BULLETIN, vol. 1, 1894, p. 63.

sible instances in which the order of T is 8. Hence we observe that the required group is the well known simple group of order 168.

It is known that there are abelian groups which cannot be groups of cogredient isomorphisms.* Suppose that the group of cogredient isomorphisms H of a given group G is Hamiltonian. Each of the operators of G which corresponds to identity of H is self-conjugate in G. Since the non-identity commutator of H is some power of each one of a set of generators of H, all the operators of G which correspond to it must also be self-conjugate in G. As this is clearly impossible, it follows that *a Hamiltonian group cannot be a group of cogredient isomorphisms*. . In fact, the preceding arguments apply to all groups which contain an operator different from identity, which is some power of each one of a set of generating operators. Hence such a group cannot be the group of cogredient isomorphisms of any group whatsoever.

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LOBACHEVSKY'S GEOMETRY.

Nikolaj Iwanowitsch Lobatschefskij. Zwei Geometrische Abhandlungen aus dem Russischen übersetzt, mit Anmerkungen und mit einer Biographie des Verfassers von FRIEDRICH ENGEL. Leipzig, B. G. Teubner, 1899. 8vo. xvi + 476 pp.

This volume is the first in the series of the "Urkunden zur Geschichte der Nichteuklidischen Geometrie" planned by Friedrich Engel and Paul Stäckel in continuation of their "Theorie der Parallellinien von Euklid bis auf Gauss." The present work on Lobachevsky has been undertaken by Engel, while a volume on the two Bolyais by Stäckel is soon to appear. All geometricians will welcome the opportunity thus afforded of becoming fully acquainted with the writings of these men, who share with Gauss the honor of being the first to break loose from the authority of Euclid and to show that the latter's axiom concerning parallel lines is not only incapable of proof but entirely unnecessary in a logical geometry which fully explains all facts of experience. The earlier works of Lobachevsky have been until now locked up in the original Russian and their contents

**Comptes rendus*, vol. 130, 1900, p. 316.

are known to the public mainly through later abstracts in French and German. Of these the best known is the small pamphlet published in Berlin in 1840 with the title "Geometrische Untersuchungen zur Theorie der Parallellinien" and translated into English by Professor G. B. Halsted. This, though containing the outline of the author's theory, is too concise to do it justice. Professor Engel has rendered therefore a signal service to mathematicians in translating the two articles which form the first part of the present volume.

The titles of these articles are, in the German translation, "Ueber die Anfangsgründe der Geometrie" and "Neue Anfangsgründe der Geometrie mit einer vollständigen Theorie der Parallellinien." Both articles appeared in publications of the Kasan University, the former in 1829–30, the latter in 1836. A footnote by Lobachevsky himself informs us that the first article is taken from one presented to the Section for Mathematical-Physical Science in 1826 under the title, "Exposition succincte des principes de la Géométrie," but no copy of this earlier work is extant. The "Anfangsgründe" is thus the earliest of Lobachevsky's publications on this subject. It is comparatively short, only sixty-six pages in the translation, and correspondingly concise, the proofs of the earlier theorems not being given. It is to the "Neue Anfangsgründe" that one must turn for the best introduction to Lobachevsky's thought. We follow this in our review.

The first six chapters present a development of geometric theorems as far as these do not depend upon any theory of parallel lines, and can be read with approval by the most orthodox believer in the euclidean geometry. Lobachevsky's order of development is however entirely new. With him the distinctive geometric property of bodies is that of contact. This idea we get through the senses, back of it we can not go; in fact, it is impossible to define it fully in words. By means of this property he discusses various kinds of sections of a body, and derives finally the conceptions of surface, line, and point, from the ways in which bodies or sections of bodies may touch each other. The distance of two points is their relative position and is determined by the contact of two bodies in the two points, independently of the form of these bodies outside of their points of contact. Hence, for example, a pair of dividers may determine distance and the idea of distance is independent of the conception of a straight line. The sphere is therefore the simplest solid, the spherical surface the sim-

plest surface. A plane is determined by the intersections of equal spheres described about two fixed points as centers, and any two of these equal spheres define a circle. A straight line is defined as a line which remains in coincidence with itself in all positions when two points are fixed, and it is shown that in any circle there exist such lines, the diameters. With these ideas as the fundamental ones, it is clear that the geometry on the sphere is as simple as the geometry on the plane, and it is not surprising to find Lobachevsky carrying the two side by side. In one respect, at least, the spherical geometry is the simpler, for it may be shown that the sum of the angles of a spherical triangle is greater than π. In the plane geometry one must content himself with the theorems that the sum of the angles of a triangle cannot exceed π, and that if the sum for one triangle is equal to π the same is true of all triangles, while if the sum is less than π it is variable, decreasing as the lengths of the sides increase. At the outset Lobachevsky assumes that space is infinite. He is therefore able to prove that planes and straight lines are infinite in extent, that two straight lines can intersect in not more than one point, and that two perpendiculars to the same straight line cannot intersect. He misses, therefore, the elliptic geometry in which the sum of the angles of a triangle is greater than π. This geometry exists for him of course upon the sphere, and the formulas which he obtains later for plane triangles are shown by him to include the spherical formulas, but the possibility of the existence of this geometry on the plane is explicitly denied. It remained for Riemann to take the last step in the freeing of geometry from unnecessary hypotheses.

In chapter VII. begins the discussion of parallel lines on which the fame of the author rests. An axiom of Euclid asserts that if a straight line intersect two straight lines forming two interior angles on the same side of the first straight line which are together less than two right angles, then the two straight lines if sufficiently produced will meet. Many attempts have been made to justify this axiom by showing that its denial involves contradictions; but all such attempts have failed. Lobachevsky boldly denies the truth of the axiom and asserts that all lines radiating from a point fall into two classes with reference to any fixed line; namely, the converging lines which meet the fixed line, and the diverging lines which do not meet it. To the latter class belong two parallel lines which form the boundary between the two classes. More precisely, if C is a point at a perpendicular distance a from a line AB, then there exists

a certain angle $\Pi(a)$ such that all lines which make with the perpendicular on the side toward AB an angle less than $\Pi(a)$ will meet AB, while lines which make an angle equal or greater than $\Pi(a)$ will not meet AB. The two lines which make the angle $\Pi(a)$ with the perpendicular are the parallel lines, and the angle $\Pi(a)$ is the angle of parallelism. If $\Pi(a)$ is identically equal to $\frac{1}{2}\pi$, the euclidean axiom results and it is readily shown that then the sum of the angles of a triangle equals π, and conversely. On the other hand, if the sum of the angles of a triangle is less than π, $\Pi(a)$ is variable, decreasing as a increases, becoming $\frac{1}{2}\pi$ only for $a = 0$, and being equal to any given acute angle for some value of a. There result therefore two systems of geometry, the usual one, and the other, called rather unfortunately by Lobachevsky the "imaginary" geometry. In the ordinary geometrya circle approaches a straight line when its radius is indefinitely increased, and the revolution of a straight line about an axis perpendicular to it generates a plane. In the imaginary geometry the limit of a circle is a boundary curve characterized by the property that all normals are parallel to a fixed line and hence to each other. The revolution of a boundary curve about a normal generates a boundary surface, which is cut by any plane in either a boundary curve or a circle. If now a triangle is formed on a boundary surface by means of three boundary curves it may be shown that the sum of its angles equals π. This triangle plays in the imaginary geometry the rôle of the plane triangle in the usual geometry.

This closes the synthetic and more elementary part of Lobachevsky's work. He turns now to the analytic treatment of the imaginary geometry. The usual trigonometric formulas hold for a boundary triangle, and a point of attack is thus found for the determination of $\Pi(a)$. It results that the angle of parallelism is an analytic function of its argument, expressed by the simple relation

$$\tan \tfrac{1}{2}\Pi(x) = e^{-x}.$$

Formulas connecting the sides and angles of a triangle are next deduced. These formulas include those of the usual plane and spherical geometry, and are in fact exactly those which one obtains by writing the formulas of spherical trigonometry for a triangle with pure imaginary sides. To derive the formulas of the euclidean geometry it is only necessary to assume that the sides of the triangle are so small that all higher powers may be neglected. This is to

assume that the world of our experience is very small in comparison with the unit of length used in the imaginary geometry.

The foregoing is a brief sketch of the prominent ideas of the "Neue Anfangsgründe." Turning back to the "Anfangsgründe" we find in addition a large number of applications to the determination of the lengths of curves, the areas of plane and curved surfaces, and the volumes of solids. Many readers who are already familiar with Lobachevsky's theory of parallels will find this part of the work of great interest; for it is not contained in any other of the author's works. There is also a discussion based on the parallax of fixed stars to show that the discrepancy between the usual and the imaginary geometry is too small to be detected experimentally.

The second part of the volume before us is occupied with notes and a life of Lobachevsky by the translator. Lobachevsky's argument is in its large outline simple and clear and the same is true in general of his separate theorems, but he is in places obscure through too concise presentation or through suppression of intermediate reckoning. Occasionally also a demonstration is faulty. All such places have been carefully worked over by the annotator with the object and the result of making the work easily read by all. The notes are more copious in the case of the "Anfangsgründe" where the need is greater.

The biography is at the same time an interesting account of Lobachevsky's career as student, professor, and rector of the Kasan University, and an historical and critical discussion of his mathematical writings. Of particular interest is the chapter on the relation of Gauss, Lobachevsky, and Bolyai, and the conclusion reached by the author that each discovered the non-euclidean geometry independently of the others. Lobachevsky was connected with Gauss through his teacher Bartels who was a friend of Gauss, and from this fact it has been argued that Lobachevsky's discoveries were inspired by Gauss. Against this theory, Professor Engel presents facts to show that Lobachevsky could have learned from Bartels nothing more than that Gauss questioned the correctness of the euclidean axiom. Lobachevsky's own writings however do not bear out the hypothesis that he learned even this much from his teacher, for the gradual development of his thought is apparent in his unpublished works, from the time when in 1816 he attempted the proof of the disputed axiom, through the period when he recognized the failure of all attempts at such a

proof but had not yet anything to take its place, until in 1826 he had found his way successfully through all his difficulties. He adds his own testimony as to the origin of his great theories in the opening sentences of the "Neue Anfangsgründe" in which he declares that the futility of the efforts made during two thousand years since Euclid to complete the theory of parallel lines aroused in him the suspicion that the ideas sought to be proved were not necessarily true. While it is remarkable that the solution of a two-thousand year old problem should be given almost simultaneously by three men, it should be remembered that these three were not the only mathematicians who had worked upon the problem. More than one had missed the solution by a hairsbreadth only; Lobachevsky, Bolyai, and Gauss succeeded in finding it.

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VOGT'S ALGEBRAIC SOLUTION OF EQUATIONS.

Leçons sur la Résolution algébrique des Équations. Par H. VOGT, professeur adjoint à la Faculté des Sciences de Nancy. Paris, Nony et Cie., 1895. 8vo., viii+201 pp.

THE present work is, we suppose, intended to be an introduction to the *modern* theory of the algebraic solution of equations. It is true that the word *modern* does not appear in the title, but however elementary the character of a *new* book of this kind may be, it is natural to suppose that the author will present his material in accordance with modern points of view, as far as these are elementary and simple.

This, however, is not the case with the volume in hand, as we proceed to show. First and foremost we have the following serious criticism to make. The rockbed of the modern theory of the algebraic solution of equations is the principles of Galois. A text book on this subject which does not explain these with all detail and use them systematically from start to finish cannot be called modern.

That the present volume sins grievously in this respect can be shown at once. Galois' theory proposes a perfectly simple and uniform scheme for the solution of any given equation. In a work of this kind this scheme should be developed at the start and then undeviatingly employed

throughout the work whenever the solution of a particular class of equations is being effected. Almost the first step in this scheme relates to the adjunction of a rational function of the roots of the given equation and the determination of the ensuing Galoisian group. Is it possible to fancy a modern treatment of the subject this book deals with, which takes up this problem, as fundamental as it is simple, just ten pages before its close! Such a fact means simply that the author throws Galois' theory to the winds.

The reason for this is nowhere explicitly given. It cannot be that Galois' principles are not fruitful; for they occupy a central position in the great field of modern mathematical speculation. It cannot be because they are too difficult to be treated in an elementary text book; for as we shall see the author treats questions much more abstruse than the elementary Galoisian theory offers. It cannot be that the problems it enables us to solve are uninteresting; for every one familiar with Galois' theory knows the contrary. It may be that M. Vogt thinks Galois' methods are not all rigorous. Chapter IX. at least lends color to this supposition. This chapter treats of the algebraic solution of equations and culminates in the theorem of Ruffini and Abel that the general equation of degree greater than four cannot be solved algebraically. The author frankly states that it is taken *en grande partie* from Chapter XIII. of Netto's book on Substitutionentheorie und ihre Anwendungen auf die Algebra. Now Netto in this chapter takes a stand which is either trivial or which is an impugnment of the correctness of those parts of Galois' theory which treat of problems of this character. As the question is altogether fundamental and as put by Netto may easily lead one astray, it is worth spending a few words on it. The question at issue is this: Does Galois' theory enable us to draw into the circle of our reasoning irrational resolvents, *i. e.*, resolvents whose roots are not rational functions of the roots of the given equation, or does it not? The statement of Netto's in this connection is this :* since the theory of substitutions treats only of *rational* functions of the roots it is impossible to employ this theory when dealing with irrational functions. To use the theory of substitutions, then, in a problem which requires the consideration of irrational resolvents would be a *petitio principii*. Such problems he declares can be settled only by algebraic reasoning. Strictly speaking this is doubtless quite correct, but taken in the strict sense these remarks are trivial. If taken in the sense that the

* L. c., p. 235.

demonstration, for example, of the Ruffini-Abel theorem by the Galoisian theory is unsound, and that it must be replaced by the laborious algebraical reasoning which Netto employs* and in which he is followed by Vogt, Netto's position is incorrect and pernicious. That irrational resolvents can be employed directly in Galois' theory follows from the simple fact that, whenever a reduction of the Galoisian group takes place on adjoining an accessory irrationality, the same effect can be produced by a rational function of the roots. The theorem which is fundamental in these questions, and which cannot be insisted on too much, is due to Jordan and given in his Traité, p. 269. The demonstration of this theorem is entirely substitution-theoretical. How does this agree with Netto's declaration: "treten daher * * * irrationale Funktionen der Wurzeln auf, so befinden wir uns auf einem Gebiet, in dem von Substitutionen überhaupt keine Rede mehr sein kann." It would certainly enlighten many if Professor Netto would explain how these remarks are to be put in harmony with §§ 230, 231 of his book, particularly with the statement made at the close of § 230.

Let us look now at the selection of material M. Vogt has to offer the reader. In a work that treats Galois' theory so shabbily we are not surprised to find only some twenty pages devoted to the theory of substitution groups. This is certainly unfortunate. The theory of groups is every day winning in importance. One has only to think of the rôle they play in geometry and in the theory of differential equations. The Galoisian theory of equations offers a splendid opportunity to introduce the student to an important part of this great theory, namely, the theory of *finite* groups. At the very start of the Galoisian theory, substitution groups demand our attention. The notions of transitivity, primitivity, invariant subgroups, series of composition, and isomorphism present themselves simply and naturally at once. A little later when we begin to consider more carefully the groups of the resolvents we find it necessary to pass from the narrow notion of a substitution group to the broad and fertile notion of a group in the abstract. A little later still we come to the groups of the regular bodies, *i. e.*, to particular cases of the all important linear group. It is with regret that we see such an opportunity entirely ignored.

Instead then of a treatment of such questions, we find a wearisome reproduction of some of Kronecker's algebraical

* It is Kronecker's modification of Abel's proof. Cf. *Monatsber. d. Berl. Akad.*, 1879, p. 205.

theories. A chapter is devoted to cyclic and metacylic functions of n independent variables; another chapter presents Kronecker's celebrated method of decomposing a form in n variables for an arbitrary domain of rationality. A third chapter gives Kronecker's treatment of abelian equations; and a fourth, Kronecker's researches on metacylic equations of prime degree. We do not wish to be understood as underestimating Kronecker's methods; on the contrary, we are an ardent admirer of them. But the methods of Kronecker here given form but a small part of the equipment of this mathematical Hercules. To try to give what is necessary would utterly crush the reader; to give no more than M. Vogt has given seems to us wholly inadequate. With half the space, all the results of importance the author has given could be demonstrated by Galois' theory and with far greater ease to the reader.

I pass now to a few criticisms of detail. The systematic employment of indeterminates should certainly be preceded by a few words of explanation. The author does not bring out with sufficient emphasis the meaning of the very fundamental terms: valeur numérique, valeurs numériquement distinctes, algébriquement distinctes. The author seems to be influenced by Kronecker's dictum which forbids the use of purely logical definitions. At least in two important instances he has followed it, viz., in the decomposition of a form into irreducible factors, and in the actual determination of the Galoisian group for a given equation. A third equally important case he has not treated, viz., the problem of determining whether a given rational function of the roots belongs to a given group or not.

On p. 62 we are given a definition of a general equation, and the remark is made that equations whose coefficients are integers are special. On p. 80 special equations are said to be those whose Galoisian group is not the symmetric group. There is a confusion of terms here. As Hilbert showed for the first time there are an infinity of equations whose group is the symmetric group and whose coefficients are integers.*

On p. 146 the rule for forming cyclic functions is not universally applicable, as simple examples show.

Consider the Abelian equation for $R(1)$, $x^4 + 1 = 0$. Let $\varepsilon = e^{2\pi i/8}$; set $x_0 = \varepsilon$, $x_1 = \varepsilon^3$, $x_2 = \varepsilon^5$, $x_3 = \varepsilon^7$; also let $x_1 = \vartheta_1 x_0$, $x_2 = \vartheta_2 x$. Then Kronecker's scheme for the roots becomes

$$\vartheta_1^{h_1}\vartheta^{h_2}x_0 \qquad (h_1, h_2 = 0, 1).$$

* Crelle, vol. 110.

The functions y_κ become here

$$y_0 = \sum_{h_2} \vartheta_2^{h_2} x_0 = x_0 + x_2; \quad y_1 = \sum_{h_1} \vartheta_1^{h_1} x_0 = x_1 + x_3.$$

Now both y_0 and y_1 are zero; they are thus not cyclic, and the rule breaks down.

Finally we observe that the treatment in Chapters X. and XII. of Kronecker's problem, of finding the necessary and sufficient form of the roots of all algebraically solvable equations of prime degree n, is far too condensed for so abtruse a matter. It is also lacking in rigor in two essential points. The question whether the functions

$$\psi_\mu = \sum_{\nu=0}^{n-1} \omega^{\mu\nu} x_\nu \qquad (\omega^n = 1)$$

vanish or whether the functions

$$y_q = \psi_{g^q} \psi_{g^{q-1}}^{-q} \qquad (q = 1, 2, \cdots, n-1)$$

are distinct is not discussed.

Before closing we beg to have it clearly understood that our criticisms have been made on the supposition that the volume in hand is to serve as an introduction to the modern theory of the algebraic solution of equations. To one who is already familiar with the elements of this theory, the present work will give much interesting and valuable information, particularly in regard to the methods peculiar to Kronecker. It may then serve in some measure as a preparation toward studying the papers of this great master.

JAMES PIERPONT.

YALE UNIVERSITY,
March, 1900.

ELEMENTS OF THE CALCULUS.

The Elements of the Differential and Integral Calculus, based on the Kurzgefasstes Lehrbuch der Differential- und Integralrechnung, von W. Nernst und A. Schönflies. By J. W. A. YOUNG and C. E. LINEBARGER. New York, D. Appleton and Co., 1900. 8vo., xvii + 410 pp.

OF the various new text-books on the calculus, this recent joint publication by a teacher of mathematics and a teacher of physics and chemistry will doubtless attract much interest, based as it is upon the German work, intended primarily for chemists, which appeared in 1896

from the pens of Professor Nernst, the celebrated exponent of the new physical chemistry, and Professor Schönflies, widely known for his investigations in pure and applied mathematics. Having taught calculus to classes of technical students and having been formerly a chemist on a state geological survey, the reviewer finds himself fully in sympathy with such a work and ready to approve even a greater number of examples and illustrations from the fields of chemistry and physics.

To the students of mathematics, astronomy, advanced physics, or technology the book is intended as a "pre-view" or general survey of the field, giving a clear insight into the fundamental principles, methods, and results, rather than the more elaborate and complex parts of the subject. The book is also offered to the general student in the belief that an elementary knowledge of the calculus should round out the mathematical career of the students seeking liberal culture. In the opinion of the reviewer the work is well adapted to those teachers or students of the natural or exact sciences who would privately acquire through an interesting and rigorous text a fair knowledge of the calculus and a view of its wide application in all domains of science.

The book is in two senses an introduction to the subject. In the first place, it presupposes only a knowledge of geometry, plane trigonometry, and the very simplest parts of algebra. The first seventy-five pages are devoted to an introduction to analytic geometry, giving as much as is needed in the later chapters. Chapters on higher algebra are inserted when needed for their applications to the calculus. Twenty pages are devoted to limits, in which the pedagogical turn of mind of the author or authors is clearly discernible. Several definitions of limits are given, graded in difficulty up to the modern treatment by means of the standard "epsilon." Seven pages on logarithms, ten pages of illustrations of partial fractions, fifty pages on infinite series, of which fifteen are devoted to the elementary algebraic theory, are given at convenient places in the text.

In the second place, the elementary character of the book may be indicated by the remark that the authors avoid the subjects curvature and elliptic integrals, so that the problem of the rectification of the ellipse and the problem of the simple pendulum are not even mentioned.

The elementary parts of the calculus are treated with admirable clearness and pedagogical insight, and yet the proofs possess all the rigor demanded by the modern methods in

analysis. Each fundamental process is first illustrated by simple examples chosen from geometry, mechanics, physics, or chemistry, and afterwards the general method is exhibited in mathematical formulæ.

The method of limits is used exclusively. In view of the strict avoidance of differentials, it would seem more pedagogical to replace the term differentiation by derivation. The earliest introduction (on p. 108) of the symbol dy/dx for the derivative might have been done with more care, giving temporarily the notation $\frac{d}{dx}(y)$ or the alternative notation $D_x y$. The reader would be spared the confusion usually attendant upon the first notation. That the authors themselves will admit the justness of this criticism is apparent from their remark on p. 172, "In the Differential Calculus we grew accustomed not to keep $\frac{d}{dx}$, the symbol for the operation of differentiation, invariably separated sharply from the quantity to which the operation was to be applied, but sometimes to write for compactness $\frac{dy}{dx}$, ··· instead of $\frac{d}{dx}y$, ···." As a matter of fact the former symbol occurs six times before the latter is (on p. 116) first introduced!

It is not easy to make students *feel* that dy/dx is not a fraction. Furthermore, of the two equivalent formulæ,

$$\frac{dy}{du}\cdot\frac{du}{dx}=\frac{dy}{dx}, \qquad \frac{d}{du}(y)\cdot\frac{d}{dx}(u)=\frac{d}{dx}(y),$$

the necessity of formal proof is not usually felt by the student in the case of the first form, but is apparent in the case of the second. Instead of the usual short proof of this formula (based upon the lemma that the limit of a product equals the product of the limits of the factors), the authors devote pp. 150–152 to a rather complicated proof. While the reviewer is accustomed to give both, he would rather omit the latter than the former.

The authors never define dx and dy as separately existing quantities,* but still retain the historic symbol $\int f(x)dx$ for

*This may be done with rigor by taking them to be *any* two quantities (infinitesimal or not) whose ratio is the derivative. For application in integral calculus, we may also define dx as an arbitrary infinitesimal increment of x and dy as that dependent function equal to the product of dx and the derivative y'. Geometrically, dx and Δy refer to the curve, while dx and dy refer to its tangent.

the function whose derivative with respect to x is $f(x)$. The step involved is certainly logical, but rather severe logic. Here again the (temporary) use of an alternative symbol, as $D_x^{-1} f(x)$, coupled with a reference to the definite integral, would bring this heroic adherence to the method of limits somewhat nearer to the plane of the student. Indeed, what student will not stumble at the statement on p. 252: "The sign $\int$ represents a form of *s* now obsolete, standing for the word *sum*, and ydx represents the type of the terms of the sum." The refusal to define dx makes the formula (p. 192) for integration by parts quite cumbrous. A readjustment must certainly take place somewhere in a student's career, if he is to continue his mathematical or physical studies in the modern literature.

On p. 296 and in the examples on p. 297 and pp. 301–307, it is required to find dy/dx when y is given as an implicit function of x, say $\varphi(x, y) = 0$. It would seem desirable to explain the usual method of taking the derivative with respect to x of both members of the equation and then solving for dy/dx. This procedure is more practical (even if it involves the calculation of the same derivatives) since it does not require the retention of a formula somewhat difficult to remember.

On p. 300, formula (5), δ is used four times for ∂.

The last chapter, a literal translation of the corresponding chapter of the German text, is highly interesting and useful. It gives a new and successful method for the differentiation or integration of functions found empirically, *i. e.*, given by a table of experimental results. In it occurs (p. 390) a confusing change of notation. It would make for clearness to omit the x introduced, but not used, in the regular text, and to explain, elsewhere than in the example given in small print, the p and θ actually used.

By this time it will be seen that the reviewer has sought in vain for very serious grounds for criticism of the text. He may therefore be permitted to congratulate the authors upon the skill with which they have completed such an interesting and valuable introduction to the calculus.

L. E. DICKSON.

THE UNIVERSITY OF TEXAS.

PASCAL'S CALCULUS OF VARIATIONS.

Die Variationsrechnung. Von E. PASCAL. Deutsch von A. SCHEPP. B. G. Teubner, Leipzig, 1899. vi. + 146 pp.

THE German translation of the "Calcolo delle variazioni" published by Ernesto Pascal in 1897 gives to American mathematicians in convenient form the best book on the calculus of variations that has, to our knowledge, appeared up to the present time. The book consists of only 150 octavo pages and presents concisely the principal facts of the subject. A valuable feature of the work will certainly be found to be the very excellent and apparently complete bibliography given in connection with brief accounts of the development of the calculus of variations. No book with which we are acquainted could be better adapted to controvert the lay opinion that everything in mathematics is exact and beyond dispute. Again and again the author calls attention to errors made by writers in this field. He often calls attention, too, to gaps which remain still to be filled in the theory, and makes the reader sometimes feel that the results which we already have rest on a rather precarious foundation. The chief fault of the book, from our point of view, is that it sacrifices simple and natural discussion to the pursuit of the end so dear to Italian mathematicians, the greatest possible generality.

The apparent purpose of the author is to give an account, absolutely rigorous as far as it goes, of the present condition of the science. That such an end is in the calculus of variations especially difficult to attain appears from the fact that the proofs are not always precise and that the author prefers often to tell us that the work given is not rigorous rather than to attempt to make it so. A serious deficiency is the almost entire lack of reference to the work of Weierstrass. This lack is, of course, intentional and doubtless due to the author's courtesy in hesitating to refer to what is to appear in Schwarz's edition of Weierstrass's works. It seems to us, however, that Pascal, with entire justice to Weierstrass, might have absorbed a little more of Weierstrass's spirit, and distinguished more carefully among the different sorts of variations employed, laying a proper emphasis on the function of each kind. The most unsatisfactory part of the book is the treatment of Jacobi's criterium. This condition is abstruse and difficult to grasp in all its beauty and ingenuity even in the simplest case. Pascal gives a purely analytical treatment of the general case. We ven-

ture to say that few readers will care to follow him through these difficult generalities. Nevertheless, this part of the book is of value, presenting as it does results which are to be found, we believe, only in various periodicals. But of the simple geometrical meaning of Jacobi's criterium, of the fact that it is a necessary as well as a sufficient condition for a maximum or a minimum, of the fact that from this criterium we may discover how long a curve continues to satisfy the conditions of the problem—not a word is said. From this we may see then that the book is not perhaps so readable as it might have been made.

We mention some points that to us seem to require criticism. A complete list would be of considerable length and would perhaps lay an undue emphasis on the faults of the work. In the opening chapter (p. 15) the author defines a variation as something "unendlich klein" and similar to a differential. This is an unsatisfactory and confusing definition, and is amusing in juxtaposition to Pascal's comment on Lagrange (p. 5). On page 16 we find the following: "In der That, wenn die beiden unendlich nahen Curven das Bestreben haben, mit einander zusammenzufallen, so ist es selbstverständlich, dass z. B. auch ihre Tangenten zusammenzufallen suchen." Such a primeval statement is utterly out of place in a book purporting to be rigorous. In the italics preceding this quotation the "variation of the derivative" is used in place of the "derivative of the variation." This slip would seem less worthy of notice if the author considered it evident, as it really is, that the two are identical. He does not, however, but proves at length two pages afterwards that they are so. The discussion of the restrictions of the "regular variation," a common term which by the way Pascal does not use, is on the whole, gratifying. He omits, however, to state what seems to us the point of chief interest, that any special variation may with perfect freedom be used to obtain a *necessary* condition for a maximum or a minimum. In pages 17–19 the author does not always make clear his assumptions as to the nature of the variation. On page 21 he says F must be assumed continuous, while two pages before he assumed that F had finite partial derivatives, an assumption certainly implying continuity. On page 39 the reasoning is purely formal, precisely that which he condemns most severely. His statement here that his process leads to the problem of maxima and minima of ordinary analysis is not accurate. His result he justifies by a footnote which admits that the text is inconclusive.

The treatment of multiple integrals though brief and incomplete is, on the whole, very good. We are glad to see several problems mentioned which are sometimes omitted in a course on this subject, such as the general problem of the calculus of variations. This Pascal calls, without sufficient reason we think, "Mayer's problem." The book closes with a discussion of the most famous problems of the calculus, *e. g.*, Newton's problem and the brachistochrone, with shorter notices of many others. The translator has added to the book some references to articles that have appeared since 1897, and indexes.

J. K. WHITTEMORE.

HARVARD UNIVERSITY.

NOTES.

THE second number of the *Transactions* of the AMERICAN MATHEMATICAL SOCIETY, which has just appeared, consists of 162 pages and contains the following articles:—"On the metric geometry of the plane n-line," by F. MORLEY; "On relative motion," by ALEXANDER S. CHESSIN; "Plane cubics and irrational covariant cubics," by HENRY S. WHITE; "A purely geometric representation of all points in the projective plane," by JULIAN LOWELL COOLIDGE; "The decomposition of the general collineation of space into three skew reflections," by EDWIN B. WILSON; "A new method of determining the differential parameters and invariants of quadratic differential quantics," by HEINRICH MASCHKE; "On the extension of Delaunay's method in the lunar theory to the general problem of planetary motion," by G. W. HILL; "On the types of linear partial differential equations of the second order in three independent variables which are unaltered by the transformations of a continuous group," by J. E. CAMPBELL.

THE INTERNATIONAL MATHEMATICAL CONGRESS AT PARIS: At the two general sessions of the congress, the following addresses will be delivered:—August 6th: "On the historiography of mathematics," by Professor M. CANTOR.—"Three Italian analysts, Betti, Brioschi, Casorati, and three ways of considering the questions of analysis; their influence," by Professor V. VOLTERRA.—August 11th: "A page from the life of Weierstrass," by Professor G. MIT-

TAG-LEFFLER.—"The rôle of intuition and logic in mathematics," by Professor H. POINCARÉ.

The work of the congress will be carried on in six sections which will meet at least once daily. These sections are:—1° Arithmetic and algebra; 2° Analysis; 3° Geometry; 4° Mechanics, mathematical physics, and celestial mechanics; 5° Bibliography and history; 6° Teaching and methods. The members are requested to submit the subjects of their communications and the section to which they will be referred, before May 1st, to Professor H. Poincaré, president of the programme committee, rue des Grands-Augustins. Paris. The committee further requests that the time for the presentation of any communication be not allowed to exceed twenty minutes. Papers may be written and read in any of the four languages, German, English, French, and Italian.

THE final international conference on a catalogue of scientific literature has been postponed from Easter to June 12th. The conference takes place in London.

To the list of works on special branches of mathematics to be issued by B. G. Teubner, as announced in the April number of the BULLETIN, there should be added the following: W. F. OSGOOD, General theory of functions; W. WIRTINGER, Algebraic functions and their integrals, and Partial differential equations.

THE eighth edition of the logarithmic tables constructed by Professor GEORGE WILLIAM JONES, of Cornell University, has recently appeared bearing the imprint of Macmillan and Company, London, and G. W. Jones, Ithaca, N. Y., 1900.

CATALOGUES of new and second-hand mathematical works have recently been published by F. Muller and Company, Doelenstraat 10, Amsterdam, Holland, and B. Pellerano, Via Gennaro Serra 20, Naples, Italy.

H. WELTER, of Paris, has issued an elaborate catalogue of periodical publications, containing a number of broken and complete sets of the various mathematical journals of the world.

THE UNIVERSITY OF CHICAGO. The following advanced courses (four or five hours weekly) in pure mathematics have been announced for the summer quarter of 1900:—By Associate Professor H. MASCHKE: Elliptic functions; Modern analytic geometry; Advanced integral calculus, I.—

By Assistant Professor J. W. A. YOUNG: Theory of numbers; Pedagogy of mathematics (one-half course).—By Mr. J. A. SMITH: Theory of equations (first half of quarter, ten hours weekly.)

The mathematical club, with fortnightly meetings, is under the direction of the departmental faculties of mathematics and mathematical astronomy.

COLUMBIA UNIVERSITY. The following advanced courses are offered during the academic year 1900–1901 by the departments of mathematics and mechanics, each course extending throughout the year:—By Professor F. N. COLE: Riemann's theory of functions, including elliptic functions, three hours.—By Professor T. S. FISKE: Advanced calculus, three hours; Theory of abelian functions, three hours.—By Professor M. I. PUPIN: Electromagnetic theory of light, two hours; Spherical harmonics, two hours.—By Professor R. S. WOODWARD: Advanced analytical mechanics, two hours; Theory of the potential function, two hours.—By Mr. C. J. KEYSER: Modern theories of geometry, three hours.—By Dr. J. MACLAY: Analytical theory of curves of double curvature and curved surfaces, three hours.—By Mr. H. B. MITCHELL: Differential equations, three hours.—By Mr. J. C. PFISTER: Theoretical mechanics, two hours.

The mathematical colloquium is held fortnightly.

CORNELL UNIVERSITY. During the academic year 1900–1901, the following advanced courses will be offered by the department of mathematics, each course occupying either two or three hours a week, as indicated, and continuing through the year:—By Professor L. A. WAIT: Advanced analytic geometry, (a) plane, three hours, (b) solid, two hours; Advanced differential calculus, three hours.—By Professor J. MCMAHON: Theoretical mechanics, two hours; quaternions, two hours; Potential and spherical harmonics, two hours; Mathematical theory of sound, two hours.—By Professor J. H. TANNER: Algebraic invariants, two hours. By Dr. D. A. MURRAY: Differential equations, three hours; Finite differences, two hours.—By Dr. J. I. HUTCHINSON: Advanced integral calculus, two hours; Calculus of variations, three hours.—By Dr. V. SNYDER: Projective geometry, three hours; Algebraic curves, two hours; General function theory, three hours; Theory of surfaces, two hours.—By Dr. G. A. MILLER: Theory of groups of a finite order, three hours; Theory of numbers, two hours.

The Oliver mathematical club will meet every other week.

HARVARD UNIVERSITY. During the year 1900–1901 Professor BÔCHER will be absent in Europe on leave. The following advanced courses in mathematics are offered :—By Professor J. M. PEIRCE: Tetrahedral coördinates, with applications to the study of points, lines, planes, and quadric surfaces; Quaternions (second course); Quaternion imaginaries and other selected topics in quaternions; † Linear associative algebra.†—By Professors BYERLY and B. O. PEIRCE: Trigonometric series, spherical harmonics, and potential function.—By Professor B. O. PEIRCE: Elasticity.†—By Professor OSGOOD: Infinite series and products; † Galois's theory; † either Calculus of variations, or Elliptic functions.—By Dr. BOUTON: Modern geometry; Theory of functions (first course); Introduction to Lie's theory of groups.—By Mr. WHITTEMORE: Theory of equations and introduction to invariants; † Calculus (second course); Differential geometry.—By Mr. COOLIDGE: Geometry of position.

These courses will each involve three lectures a week throughout the year, except those marked †, which involve about half this number of lectures. Professors ASAPH HALL and OSGOOD also offer courses in reading and research on Selected topics in celestial mechanics and Higher analysis, respectively.

During the summer semester, 1900, the several universities below offer the following mathematical courses:

UNIVERSITY OF INNSBRUCK. By Professor OTTO STOLZ: Differential and integral calculus, with exercises in seminar, four hours; General arithmetic, second part, introduction to the theory of functions of complex variables, with exercises in seminar, three hours.—By Professor W. WIRTINGER: Analytical geometry of space (continued), three hours; Theory of numbers (continued), two hours; Seminar, two hours.

UNIVERSITY OF PRAGUE. By Professor G. PICK: Differential and integral calculus, three hours; Elements of the calculus of variations, two hours; Geometrical exercises, two hours.—By Professor F. LIPPICH: Theoretical mechanics, three hours; Theory and applications of potential, two hours.

UNIVERSITY OF VIENNA. By Professor G. VON ESCHERICH: Elements of differential and integral calculus II, five hours; Exercises on the preceding course, two hours; Seminar, two hours; Proseminar, one hour.—By Professor L. GEGEN-

BAUER: Algebra (continued), four hours; Elements of the theory of invariants, one hour; Mathematical statistics, three hours; Proseminar, one hour; Seminar, two hours.—By Professor F. MERTENS: The laws of reciprocity (continuation), five hours; Theory of probabilities, three hours; Seminar, two hours; Proseminar, one hour.—By Professor G. KOHN: Analytical geometry of space, four hours; Exercises on the preceding, one hour.—By Dr. V. SERSAWY: The mathematics of insurance, I, three hours, II, four hours.—By Dr. A. TAUBER: Elements of perspective, two hours; Insurance mathematics (continued), four hours; Exercises on the latter course, two hours.—By Dr. K. ZINDLER: Kinematics, two hours.—By Dr. E. BLASCHKE: Introduction to mathematical statistics, II, three hours.—By Dr. K. ZSIGMONDY: The transcendentality of the numbers e and π, one hour.—By Dr. R. D. v. STERNECK: Elementary theory of numbers, two hours.

THE following subject has been announced by the Madrid academy of sciences for its mathematical prize for the year 1901: "Didactic exposition of the modern theories of non-euclidean geometry, or a detailed analysis of the principal works upon this branch of science from the time of Gauss to the present." Competing memoirs should be written in Spanish or Latin and will be received by the secretary of the academy before December 31st, 1901. The first prize consists of fifteen hundred pesetas and a gold medal.

AMONG the corresponding members elected by the Berlin academy of sciences at its recent celebration commemorating the two hundredth anniversary of its foundation are the mathematicians Professor P. GORDAN, of Erlangen, Professor F. MERTENS, of Vienna, and Professor F. SCHOTTKY, of Marburg.

PROFESSOR P. DRUDE, of Leipsic, has become professor of mathematical physics at the University of Giessen.

IT is reported that Professor L. BOLZMANN, of the University of Vienna, has received a call to the University of Leipzig and is considering it favorably.

DR. J. HORN, formerly docent at the Polytechnic Institute at Charlottenburg, has been called to the School of Mines at Clausthal as professor of mathematics.

DR. L. E. DICKSON has resigned an associate professorship in mathematics at the University of Texas to become assistant professor at the University of Chicago. DR. H.

E. SLAUGHT has been promoted to an assistant professorship of collegiate mathematics in the latter university.

UNDER the recent readjustment of its relation to Columbia University, Barnard College is hereafter to have a distinct department of undergraduate mathematics. Professor F. N. COLE has been made head of this department, retaining however his position in the university graduate school of pure science. Dr. EDWARD KASNER (Ph.D., Columbia) has been appointed tutor, and Miss GRACE ANDREWS (M.A., Columbia) assistant in the department at Barnard.

DR. J. B. CHITTENDEN, tutor in mathematics at Columbia University, has been appointed professor of mathematics in the Brooklyn Polytechnic Institute, Brooklyn, N. Y. Mr. H. B. MITCHELL, assistant in mathematics at Columbia, has been promoted to a tutorship.

MR. A. D. Fox has been made professor of mathematics at Whitman College, Walla Walla, Wash.

MR. E. A. PATTINGILL, recently a graduate student at Cornell University, has been appointed instructor in mathematics in the Iowa State Agricultural College.

THE death is announced of Professor B. CHRISTOFFEL, professor of mathematics at the University of Strassburg, aged seventy years; Professor Christoffel had offered no lectures since 1894 on account of ill health.

THE death is also announced of Professor WILLIAM M. THRASHER, for many years professor of mathematics at Butler University, Irvington, Ind.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BENNER (H.). Bestimmung der Coefficienten, welche bei der Berechnung der Integrale $\int \frac{x^n dx}{\sqrt{1+ax+bx^2}}$ und $\int \frac{x^n dx}{\sqrt{1+ax+bx^2+cx^3}}$ auftreten. (Diss., Erlangen, 1897.) Boston, Ginn, 1899. 8vo. 10 + 59 pp.

BIBLIOTHECA MATHEMATICA. Zeitschrift für Geschichte der mathematischen Wissenschaften, herausgegeben von G. Eneström. 3te [erweiterte] Folge. Vol. I: 1900. Leipzig, Teubner, 1900. Per annum M. 20.00

BÖGER (R.). Ebene Geometrie der Lage. Leipzig, Göschen, 1900. 8vo. 10 + 289 pp. Cloth. (Sammlung Schubert, No. VII.) M. 5.00

BROWN (E. W.). See TRANSACTIONS.

BULLART (W. G.) On the general classification of plane quartic curves. (Diss.) Worcester, Mass., 1899. 8vo. 16 pp., 2 plates.

DELAUNAY (N.). See WASSILIEF (A.).

DUDENSING (W.). Ueber die durch eine allgemeine dreigliedrige algebraische Gleichung definierte Funktion und ihre Bedeutung für die Auflösung der algebraischen Gleichungen von höherem als viertem Grade. Leipzig, Teubner, 1900. 8vo. 8 + 52 pp. M. 2.40

ENESTRÖM (G.). See BIBLIOTHECA.

FELDBLUM (M.). Ueber elementar-geometrische Konstruktionen. (Diss.) Göttingen, 1899. 8vo. 48 pp., 1 plate.

FISKE (T. S.). See TRANSACTIONS.

FREGE (G.). Ueber die Zahlen des Herrn H. Schubert. Jena, Pohle, 1900. 8vo. 6 + 32 pp. M. 1.20

GANTER (H.) und RUDIO (F.). Die Elemente der analytischen Geometrie. Zum Gebrauche an höheren Lehranstalten, sowie zum Selbststudium. Mit zahlreichen Uebungsbeispielen. Teil I: Die analytische Geometrie der Ebene. 4te Auflage. Leipzig, Teubner, 1900. 8vo. 8 + 180 pp. Cloth. M. 3.00

GASCO (D. L. G.). See MANSION (P.).

GRAF (J. H.) und GUBLER (E.). Einleitung in die Theorie der Besselschen Funktionen. Heft 2: Die Besselsche Funktion zweiter Art. Bern, Wyss, 1900. 8vo. 7 + 156 pp. M. 3.20

HAGEN (J. G.). Synopsis der höheren Mathematik. (In 4 Bänden.) Vol. III: Differential- und Integralrechnung. (In circa 5 Lieferungen.) Berlin, 1900. 4to. Lieferung 1: pp. 1–64. M. 5.00.

JAHRBUCH über die Fortschritte der Mathematik, begründet von C. Ohrtmann. Herausgegeben von E. Lampe. Vol. 28, Jahrgang 1897 3tes (Schluss-) Heft. Berlin, Reimer, 1900. 8vo. 68 pp. + pp 609–935. M. 12.40

KNESER (A.). Lehrbuch der Variationsrechnung. Braunschweig, Vieweg, 1900. 8vo. 15 + 311 pp. M. 8.00

KÖLMEL (F.). Bewegungen und Umlegungen der Ebene bei projektiver Massbestimmung. Untersuchungen zur nichteuklidischen Geometrie. Lahr, Schauenburg, 1900. 8vo. 7 + 99 pp., 1 plate. M. 2.50

LAMPE (E.). See JAHRBUCH.

LAURENT (H.). Cours de mathématiques, professé à l'Institut agronomique. Paris, Carré et Naud, 1900. 8vo. 224 pp.

MANSION (P.). Introduction à la théorie des déterminants, avec de nombreux exercices, à l'usage des établissements d'instruction moyenne. 3e édition. Gand, Hoste, 1899. 8vo. 40 pp. Fr. 1.00

——. Eléments de la théorie des déterminants, avec de nombreux exercices. 6e édition, revue et augmentée. Paris, Gauthier-Villars, 1900. 8vo. 6 + 91 pp. Fr. 3.00

——. Elemente der Theorie der Determinanten. Mit vielen Uebungsaufgaben. 3te, vermehrte Auflage. Leipzig, Teubner, 1900. 8vo. 103 pp. M. 2.60

——. Theoria sucinta de las funciones hiperbolicas. Traducido por D. L. G. Gasco. Valencia, Aguilar, 1899. 8vo. 44 pp. Fr. 1.00

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MOORE (E. H.). See TRANSACTIONS.

NETTO (E.). Vorlesungen über Algebra. Vol. 2, 2te (Schluss-) Lieferung. Leipzig, Teubner, 1900. 8vo. 12 pp. + pp. 193–519. M. 10.00.

RICE (H. L.). The theory and practice of interpolation, including mechanical quadrature, and other important problems concerned with the tabular values of functions. With the requisite tables. Lynn, Mass., Nichols, 1899. Folio. 10 + 234 pp. Cloth.

RUDIO (F.). See GANTER (H.).

SIMON (M.). Analytische Geometrie der Ebene. Leipzig, Göschen, 1900. 8vo. 7 + 372 pp. Cloth. (Sammlung Schubert, No. VIII.) M. 6.00

TRANSACTIONS of the American Mathematical Society. Edited by E. H. Moore, E. W. Brown, T. S. Fiske. Published quarterly by the Society with the coöperation of Harvard University, Yale University, Princeton University, Columbia University, Haverford College, Northwestern University, Cornell University, The University of California, Bryn Mawr College, The University of Chicago. Vol. I, No. 1, January, 1900. New York, The Macmillan Company, 1900. 4to. 96 pp. Per annum $5.00

VIVANTI (G.). See MEYER (W. F.).

WASSILIEF (A.) und DELAUNAY (N.). P. L. Tschebyschef. (P. L. Tschebyschef und seine wissenschaftlichen Leistungen, von Wassilief; Die Tschebyschefschen Arbeiten in der Theorie der Gelenkmechanismen, von Delaunay.) Leipzig, Teubner, 1900. 8vo. 3 + 70 pp., portrait. M. 4.00

II. ELEMENTARY MATHEMATICS.

BAGNOLI (E.). Geometria rettilinea e curvilinea, trattata con metodo preeuclidico, e cronogoniometria. Roma, Loescher, 1899. 8vo. 295 pp., 24 plates. Fr. 5.00

——. Trattato delle corde nel circolo. Roma, Loescher, 1899. 8vo. 81 pp., 10 plates. Fr. 3.00

BEHR (A.). Die Planimetrie (ebene Geometrie). Leipzig, Schäfer, 1900. 8vo. 8 + 104 pp. (Die Schule des Bautechnikers, herausgegeben im Verein mit Lehrern an Bau- und anderen technischen Fachschulen von F. Stade, Vol. 2.) M. 2.00

BOHNERT (F.). Ebene und sphärische Trigonometrie. Leipzig, Göschen, 1900. 8vo. 8 + 160 pp. Cloth. (Sammlung Schubert, No. III.) M. 2.00

CHAILAN (E.). Eléments de trigonométrie, à l'usage de la classe de seconde moderne. Paris, Poussielgue, 1900. 16mo. 64 pp.

EHRHARDT (H.). Neues System der Flächenberechnung und Flächenteilung mit Hülfe einer planimetrischen Tafel, welche zugleich als Produkten- und Quadrattafel dient, nebst einer Sinustafel. Stuttgart, 1900. 8vo. 91 + 9 pp., 3 plates. M. 3.00

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KUHN (K.). Lehrbuch der Elementar-Arithmetik. Teil 1. 2te Auflage. Hildburghausen, Pezoldt, 1900. 8vo. 4 + 48 pp. (Technische Lehrhefte: Mathematik, Heft 1.) M. 1.50

LEVI (F.). Problemi elementari relativi ad alcuni casi di equivalenza di figure piane. Lodi, Dell'Avo, 1899. 8vo. 23 pp., 2 plates. Fr. 0.50

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RAJNA (M.). MÜLLER (O.).

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SOLUCIONES y respuestas de los ejercicios y problemas contenidos en las Lecciones elementales de aritmética, por los Hermanos de las escuelas cristianas. Libro del maestro. 3a edicion, corregida. Paris, Oudinot, 1900. 16mo. 136 pp.

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UTH (K.). Planimetrie. Leitfaden mit Konstruktionsaufgaben und Uebungssätzen. 6te Auflage, von R. Franz. Kassel, Hühn, 1900. 8vo. 8 + 157 pp. Cloth. M. 2.00.

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III. APPLIED MATHEMATICS.

ALMY (J. E.). Ueber die Entladungspotentiale in festen und tropfbarflüssigen Dielektricis. (Diss.) Berlin, Mayer & Müller, 1900. 8vo. 35 pp. M. 1.20

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KUGLER (F. X.). Die babylonische Mondrechnung. Zwei Systeme der Chaldäer über den Lauf des Mondes und der Sonne. Auf Grund mehrerer von J. N. Strassmaier, S. J., kopierten Keilinschriften des britischen Museums. Mit einem Anhang über chaldäische Planetentafeln. Freiburg i. B., Herder, 1900. 8vo. 15 + 215 pp., 13 plates. M. 24.00

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MASONI (U.). Corso di idraulica teorica e pratica. 2a edizione, notevolmente ampliata. Napoli, 1899. 8vo. 24 + 680 pp. Fr. 9.50

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